

# Energy-Optimal Motion Planning for Agents: Barycentric Motion and Collision Avoidance Constraints

Logan E. Beaver, *Student Member, IEEE*, Michael Dorothy,  
Christopher Kroninger, Andreas A. Malikopoulos, *Senior Member, IEEE*

**Abstract**—As robotic swarm systems emerge, it is increasingly important to provide strong guarantees on energy consumption and safety to maximize system performance. One approach to achieve these guarantees is through constraint-driven control, where agents seek to minimize energy consumption subject to a set of safety and task constraints. In this paper, we provide an equivalent sufficient and necessary optimality condition for an energy-minimizing agent with integrator dynamics that only depends on the state and control actions of the agent. In particular, we show that the agent must have a continuous control input at the transition between unconstrained and constrained trajectories. In addition, we present and analyze barycentric motion and collision avoidance constraints to be used in constraint-driven control of swarms.

## I. INTRODUCTION

Control of swarms systems is an emerging topic in the fields of controls and robotics. Due to their adaptability and flexibility [1], swarm systems have attracted considerable attention in transportation [2], construction [3], and surveillance [4] applications. As we advance to experimental swarm testbeds [5], [6] and outdoor experiments [7], it is critical to minimize the cost per agent in the swarm by considering energy-minimizing algorithms with strong guarantees on safety and performance.

Our main contribution is motivated by the need to find real-time solutions to constrained optimal control problems. In general, these problems require propagating the system dynamics forward in time from an initial state, then propagating the system backwards in time using the Euler-Lagrange equations. Constraints may introduce jumps and corners along the trajectory, resulting in a feasible space that is challenging to optimize over in real-time. The main contribution of this paper is an equivalent set of optimality conditions on the control input of the system that do not involve the backward dynamics. Similar results have been explored on a case-by-case basis [8]–[10]; however, there has been no general continuity result reported in the literature. Our main result is applicable as a coarse high-level plan for many systems, particularly for planning automated vehicle [11] and drone [12] trajectories, which are then tracked by

This research was supported by Combat Capabilities Development Command, Army Research Laboratory, MD, USA.

Logan and Andreas are with the Department of Mechanical Engineering, University of Delaware, Newark, DE, USA. Michael and Christopher are with Combat Capabilities Development Command, Army Research Laboratory, MD, USA. (emails: lebeaver@udel.edu; michael.r.dorothy.civ@mail.mil; christopher.m.kroninger.civ@mail.mil; andreas@udel.edu.)

a suitable low-level controller. Our second contribution is a barycentric motion constraint inspired by the distributed formation control law presented in [13]. Our contribution is an explicit formulation of the Barycentric motion constraint that drives the agents within a fixed distance of a reference position and keeps them there. We embed this constraint in an optimal control problem and provide necessary and sufficient conditions for optimality that only depend on the state and control action of the agent.

The remainder of the paper is organized as follows. In Section II, we provide our main result that gives sufficient and necessary conditions for our proposed agent to have a continuous control input. In Section III, we propose the barycentric motion constraint, and in Section IV, we derive the corresponding optimal motion primitive. In Section IV-C, we derive the remaining optimality conditions in terms of the state and control of the agent, and in Section IV-D, we present the optimality conditions for the agent to constrain itself to the surface of a closed disk, and equivalently, collision avoidance. Finally, we draw concluding remarks and future work in Section V.

## II. MAIN RESULT

Consider a dynamical system with state space  $\mathcal{S}(t)$  containing  $k$  states in  $\mathbb{R}^n$  at time  $t \in \mathbb{R}$ ,

$$\mathcal{S}(t) = \{\mathbf{x}_1(t), \mathbf{x}_2(t), \dots, \mathbf{x}_k(t)\}, \quad (1)$$

where  $\mathbf{x}_i(t) \in \mathbb{R}^n$ ,  $i = 1, \dots, k$ ,  $k \in \mathbb{N}$ . Let the system obey integrator dynamics, i.e.,

$$\dot{\mathbf{x}}_p(t) = \begin{cases} \mathbf{x}_{p+1}(t), & \text{if } p \in \{1, 2, \dots, k-1\}, \\ \mathbf{u}(t), & \text{if } p = k, \end{cases} \quad (2)$$

where  $\mathbf{u}(t) \in \mathbb{R}^n$  is the bounded control input, i.e.,  $\|\mathbf{u}(\cdot)\| < \infty$ . Let  $e(t)$  be the rate of energy consumption of the system given by

$$e(t) = \frac{1}{2} \|\mathbf{u}(t)\|^2, \quad (3)$$

i.e., rate of energy consumption is proportional to the  $\mathcal{L}^2$  norm of the control input. Finally, we impose the following constraint,

$$g(\mathbf{x}(t), t) \leq 0, \quad (4)$$

where  $g(\mathbf{x}(t), t)$  is a class  $C^{q-1}$  function, and  $q \in \mathbb{N}$  is the minimum number of time derivatives of  $g(\mathbf{x}(t), t)$  required for any component of  $\mathbf{u}(t)$  to appear in  $g^{(q)}(\mathbf{x}(t), t)$ .

To compute the energy-optimal control input for the system we follow the standard procedure for constrained optimal

control [14]. First we take  $q - 1$  time derivatives of (4) to construct a vector of tangency equations,

$$\mathbf{N}(\mathbf{x}(t), t) = \begin{bmatrix} g(\mathbf{x}(t), t) \\ g^{(1)}(\mathbf{x}(t), t) \\ \dots \\ g^{(q-1)}(\mathbf{x}(t), t) \end{bmatrix}. \quad (5)$$

Over any nonzero time interval where the constraint is active, i.e.,  $g(\mathbf{x}(t), t) = 0$ , (5) must be equal to  $\mathbf{0}$  everywhere. To generate the optimal trajectory, we take  $g^{(q)}(\mathbf{x}(t), t)$  as a control constraint and enforce  $\mathbf{N}(\mathbf{x}(t), t) = \mathbf{0}$  when  $g(\mathbf{x}(t), t) = 0$ . Next, we seek the optimal control input  $\mathbf{u}(t)$  that minimizes the Hamiltonian,

$$H = \frac{1}{2} \|\mathbf{u}(t)\|^2 + \boldsymbol{\lambda}(t) \cdot \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) + \mu(t) g^{(q)}(\mathbf{x}(t), t), \quad (6)$$

where  $\boldsymbol{\lambda}(t) \in \mathbb{R}^{k \times n}$  are the influence functions,  $\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t))$  are the integrator dynamics defined by (2), and  $\mu(t) \in \mathbb{R}_{\geq 0}$  is a Lagrange multiplier where

$$\mu(t) \begin{cases} = 0, & \text{if } g(\mathbf{x}(t), t) < 0, \\ \geq 0, & \text{if } g(\mathbf{x}(t), t) = 0. \end{cases} \quad (7)$$

The optimal control input must satisfy  $\frac{\partial H}{\partial \mathbf{u}} = 0$ , thus the optimal unconstrained input is

$$\mathbf{u}(t) = -\boldsymbol{\lambda}_k(t), \quad (8)$$

where  $\boldsymbol{\lambda}_k(t)$  is the influence function corresponding to the state  $\mathbf{x}_k(t)$  and  $k = |\mathcal{S}(t)|$ . Next we present Lemma 1 and our main result, Theorem 1.

**Lemma 1.** Given real vectors  $\mathbf{a}, \mathbf{b} \in \mathbb{R}^m$ , the unique real solution to the equation  $\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 = 2\mathbf{a} \cdot \mathbf{b}$  is  $\mathbf{a} = \mathbf{b}$ .

The proof of Lemma 1 follows trivially from the relation  $\|\mathbf{a} - \mathbf{b}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\mathbf{a} \cdot \mathbf{b}$ .

**Theorem 1.** Consider the dynamical system with the state space (1), integrator dynamics (2), an energy cost (3), and a scalar functional constraint on the state trajectory  $g(\mathbf{x}(t), t) \leq 0$ . Suppose at a time  $t_1 \in \mathbb{R}$  the system transitions between an unconstrained and constrained trajectory, i.e.,  $g(\mathbf{x}(t_1), t_1) = 0$ ,  $\mathbf{N}(\mathbf{x}(t_1), t_1) = \mathbf{0}$ , and for  $\epsilon > 0$   $g(\mathbf{x}(t_1 - \epsilon), t_1 - \epsilon) < 0$ . If there exists  $t_2 > t_1$ ,  $t_2 \in \mathbb{R}$ , such that  $g(\mathbf{x}(t), t) = 0$  for all  $t \in [t_1, t_2]$ , and  $g^{(q)}(\mathbf{x}(t_1), t_1)$  exists, then the optimal control input  $\mathbf{u}(t_1)$  is continuous.

*Proof.* The jump conditions of the influence functions and Hamiltonian at time  $t$  are

$$\boldsymbol{\lambda}^T(t^+) = \boldsymbol{\lambda}^T(t^-) - \boldsymbol{\pi}^T \frac{\partial \mathbf{N}}{\partial \mathbf{x}} \Big|_t, \quad (9)$$

$$H(t^+) - H(t^-) = \boldsymbol{\pi}^T \frac{\partial \mathbf{N}}{\partial t} \Big|_t, \quad (10)$$

where  $t^-$  and  $t^+$  correspond to the left and right limits of  $t$ , respectively, and  $\boldsymbol{\pi}$  is a  $q \times 1$  vector of constant Lagrange multipliers. Substituting (6) into (10) yields

$$\frac{1}{2} \|\mathbf{u}^+\|^2 + \boldsymbol{\lambda}^+ \cdot \mathbf{f}^+ - \frac{1}{2} \|\mathbf{u}^-\|^2 - \boldsymbol{\lambda}^- \cdot \mathbf{f}^- = \boldsymbol{\pi}^T \mathbf{N}_t, \quad (11)$$

where the superscripts  $-$  and  $+$  correspond to variables evaluated at  $t^-$  and  $t^+$ , respectively, and  $\mathbf{N}_t$  is shorthand for the partial derivative of  $\mathbf{N}(\mathbf{x}(t), t)$  with respect to time. Note that  $\mu(t^-) = 0$  and  $g^{(q)}(\mathbf{x}(t^+), t^+) = 0$ , thus those terms do not appear in (11). Substituting (9) into (11) yields

$$\begin{aligned} \frac{1}{2} \|\mathbf{u}^+\|^2 + \left( \boldsymbol{\lambda}^- - (\boldsymbol{\pi}^T \mathbf{N}_x)^T \right) \cdot \mathbf{f}^+ - \frac{1}{2} \|\mathbf{u}^-\|^2 - \boldsymbol{\lambda}^- \cdot \mathbf{f}^- \\ = \boldsymbol{\pi}^T \mathbf{N}_t, \end{aligned} \quad (12)$$

where  $\mathbf{N}_x$  is shorthand for the partial derivative of  $\mathbf{N}(\mathbf{x}(t), t)$  with respect to  $\mathbf{x}(t)$ .

Next, we simplify the influence functions in (12) using continuity of the states and (8),

$$\begin{aligned} \boldsymbol{\lambda}^- \cdot \mathbf{f}^+ - \boldsymbol{\lambda}^- \cdot \mathbf{f}^- &= (\boldsymbol{\lambda}_1^- \cdot \mathbf{x}_2 + \dots + \boldsymbol{\lambda}_k^- \cdot \mathbf{u}^+) \\ &\quad - (\boldsymbol{\lambda}_1^- \cdot \mathbf{x}_2 + \dots + \boldsymbol{\lambda}_k^- \cdot \mathbf{u}^-) \\ &= \boldsymbol{\lambda}_k^- \cdot \mathbf{u}^+ - \boldsymbol{\lambda}_k^- \cdot \mathbf{u}^- \\ &= -\mathbf{u}^- \cdot \mathbf{u}^+ + \|\mathbf{u}^-\|^2. \end{aligned} \quad (13)$$

Substituting (13) into (12) and rearranging yields

$$\frac{1}{2} \|\mathbf{u}^+\|^2 + \frac{1}{2} \|\mathbf{u}^-\|^2 - \mathbf{u}^+ \cdot \mathbf{u}^- = \boldsymbol{\pi}^T \mathbf{N}_t + (\boldsymbol{\pi}^T \mathbf{N}_x) \mathbf{f}^+. \quad (14)$$

By Lemma 1, the control input  $\mathbf{u}(t)$  is continuous at the junction if and only if the right hand side of (14) is zero, hence we may formulate the equivalent condition,

$$\boldsymbol{\pi}^T (\mathbf{N}_t + \mathbf{N}_x \mathbf{f}^+) = 0. \quad (15)$$

Since  $\boldsymbol{\pi}$  is generally nonzero, we seek to satisfy

$$\mathbf{N}_t + \mathbf{N}_x \mathbf{f}^+ = \mathbf{0}. \quad (16)$$

Since  $g(\mathbf{x}(t), t) = 0$  for all  $t \in [t_1, t_2]$ , its  $q$  derivatives exist and are equal to zero for all  $(t_1, t_2)$ , thus  $\mathbf{N}(\mathbf{x}(t_1^+), t_1^+) = \mathbf{0}$ . Equating the time derivative of (5) at  $t_1^+$  to (16) yields the equivalent condition

$$\frac{d}{dt} \mathbf{N}(\mathbf{x}(t_1^+), t_1^+) = \mathbf{N}_t + \mathbf{N}_x \mathbf{f}^+ = \mathbf{0}. \quad (17)$$

Eq. (17) can be expanded into a system of equations for the right-hand derivative of each row  $r$ , namely

$$\frac{d}{dt} g^{(r)+} = \frac{\partial}{\partial t} g^{(r)+} + \frac{\partial g^{(r)+}}{\partial \mathbf{x}} \frac{d\mathbf{x}^+}{dt}, \quad (18)$$

where  $r = 0, 1, 2, \dots, q - 1$ . Condition (18) is the definition of right  $q$ -derivatives of  $g(\mathbf{x}, t)$ , which exist by our premise. Thus, (16) is always satisfied and the control input  $\mathbf{u}(t)$  is continuous at  $t = t_1$ .  $\square$

Note that if the system transitions back to the unconstrained case at some  $t_3 > t_2$ , then the optimal control input  $\mathbf{u}(t_3)$  is continuous. The proof is identical to that of Theorem 1 and has been omitted due to space limitations.

Theorem 1 also applies in the case that two constraints  $g_1(\mathbf{x}(t), t)$  and  $g_2(\mathbf{x}(t), t)$  activate simultaneously at time  $t_1$ . In this case the tangency equation (5) will contain the time derivatives of both constraints, and (9) - (18) hold.

**Corollary 1.** Consider the dynamical system described in Theorem 1. If the system is traveling along the trajectory

imposed by the constraint  $g(\mathbf{x}(t), t) = 0$ , and the tangency conditions,  $\mathbf{N}(\mathbf{x}(t), t)$ , are discontinuous at some time  $t_1$ , then the control input at  $\mathbf{u}(t_1)$  is continuous if and only if a feasible  $\mathbf{u}(t_1)$  exists.

*Proof.* Let  $t_1$  be the time where any element of  $\mathbf{N}(\mathbf{x}(t_1), t_1)$  is discontinuous, while  $\mathbf{N}(\mathbf{x}(t_1^-), t_1^-) = \mathbf{0}$ . Continuity in the system state implies that at least one row of  $\mathbf{N}(\mathbf{x}(t_1^+), t_1^+)$  must be nonzero. To satisfy  $g(\mathbf{x}(t), t) = 0$  for  $t > t_1$  requires an infinite impulse control input at  $t_1$  [14], which contradicts the boundedness of  $\mathbf{u}(t)$ . Thus, if a feasible control input exists, the system must transition to an unconstrained arc at  $t_1$ , hence  $\mathbf{u}(t_1)$  is continuous by Theorem 1.  $\square$

Next, we present a case study for Theorem 1 and Corollary 1 for a double-integrator system in  $\mathbb{R}^2$  under a barycentric motion constraint. This is a decentralized constraint-driven optimal control problem, where barycentric motion can be directly used to drive the agent to a location or adapted for use as a collision-avoidance constraint. Under our framework, agents don't only react to the environment's current state [15]. Instead, they plan trajectories over a horizon.

### III. PROBLEM FORMULATION

As a motivating example for Theorem 1, consider a single agent in  $\mathbb{R}^2$  with double integrator dynamics

$$\dot{\mathbf{p}}(t) = \mathbf{v}(t), \quad (19)$$

$$\dot{\mathbf{v}}(t) = \mathbf{u}(t), \quad (20)$$

where  $\mathbf{p}$  and  $\mathbf{v}$  are the agent's position and velocity, respectively. The state of the agent is given by

$$\mathbf{x}(t) = \begin{bmatrix} \mathbf{p}(t) \\ \mathbf{v}(t) \end{bmatrix}. \quad (21)$$

Let  $\mathbf{p}_r(t)$  be a time-varying reference position which the agent seeks to reach in finite time. We denote the relative distance between the agent and reference state as

$$\mathbf{r}(t) = \mathbf{p}(t) - \mathbf{p}_r(t). \quad (22)$$

**Definition 1.** For a desired aggregation distance  $D \in \mathbb{R}_{>0}$ , we define the barycentric motion constraint

$$g(\mathbf{x}(t), t) = \begin{cases} \beta(\mathbf{x}(t)) + \mathbf{r}(t) \cdot \dot{\mathbf{r}}(t), & \|\mathbf{r}(t)\| > D \\ \mathbf{r}(t) \cdot \mathbf{r}(t) - D^2, & \|\mathbf{r}(t)\| \leq D \end{cases} \leq 0, \quad (23)$$

where  $\beta(\cdot) > 0$  forces the agents to move toward the closed disk. We refer to  $\|\mathbf{r}(t)\| > D$  and  $\|\mathbf{r}(t)\| \leq D$  as Case I and Case II of  $g(\mathbf{x}(t), t)$ , respectively.

In a swarm system, the reference position may correspond to individual positions within a formation, or it may depend on agent interactions [10]. Alternatively, the agents may share a global reference position with a larger value of  $D$  to encourage aggregation.

Finally, we present a constraint-driven optimal control problem to determine the energy-optimal trajectory of the agent under the barycentric motion constraint.

**Problem 1.** The problem is formulated as follows:

$$\min_{\mathbf{u}(t)} \frac{1}{2} \int_{t^0}^{t^f} \|\mathbf{u}(t)\|^2 dt \quad (24)$$

$$\text{subject to: (19), (20), (23),} \quad (25)$$

$$\text{given: } \mathbf{p}(t^0), \mathbf{v}(t^0). \quad (26)$$

where  $[t^0, t^f] \subset \mathbb{R}$  is the planning horizon for the agent.

To solve Problem 1 we impose the following assumptions.

**Assumption 1.** The initial state of the agent,  $\mathbf{x}(t^0)$ , satisfies (23).

We impose Assumption 1 to ensure that the agent can generate a feasible trajectory at time  $t^0$ , and that Problem 1 has a nonempty feasible space. This assumption may be relaxed by extending (23) to include an additional case. However, this would add complexity to the problem without fundamentally changing our analysis.

**Assumption 2.** There are no disturbances or noise, and the agent is able to track the trajectory generated by Problem 1.

We impose Assumption 2 to analyze the agent's behavior in a deterministic setting. This assumption may be relaxed by imposing some notion of robustness to Problem 1 or introducing a suitable control barrier function for tracking.

**Assumption 3.** The reference trajectory satisfies  $\frac{d^4 \mathbf{p}_r}{dt^4} = 0$ .

Assumption 3 simplifies the analysis in Section IV. This assumption may be relaxed by carrying the term  $\frac{d^4 \mathbf{p}_r}{dt^4}$  through the final steps, which adds complexity without a significant impact on our results.

### IV. SOLUTION APPROACH

As a first step, we prove the continuity of control when transitioning between the cases of the barycentric motion constraint (Definition 1). Then, we present the remaining optimality conditions such that the agent's optimal trajectory can be computed using only state and control equations.

#### A. Properties of the Barycentric Motion Constraint

The constraint  $g(\mathbf{x}(t), t)$  has two cases (see Definition 1), Case I corresponds to a *barycentric spiral*, and Case II corresponds to a closed disk centered on the reference state. When an agent transitions between an unconstrained arc and the arc defined by  $g(\mathbf{x}(t), t) = 0$ , the control input  $\mathbf{u}(t)$  is continuous by Theorem 1. Next, we present three lemmas that describe the behavior of the agent while traveling along  $g(\mathbf{x}(t), t) = 0$ .

**Lemma 2.** If there exist  $\gamma \in \mathbb{R}_{>0}$  that lower bounds  $\beta(\mathbf{x}(t))$  for all  $t \in \mathbb{R}$ , then the agent will satisfy  $\|\mathbf{r}(t)\| \leq D$  in finite time.

*Proof.* Let the agent satisfy  $\|\mathbf{r}(t)\| > D$ . By Definition 1,  $\beta(\mathbf{x}(t)) + \mathbf{r}(t) \cdot \dot{\mathbf{r}}(t) \leq 0$ . This implies that  $\beta(\mathbf{x}(t)) + \|\mathbf{r}(t)\| \|\dot{\mathbf{r}}(t)\| \cos \theta_r(t) \leq 0$  by the definition of the dot product, where  $\theta_r(t)$  is the angle between  $\mathbf{r}(t)$  and  $\dot{\mathbf{r}}(t)$ .

Substituting the lower bounds for  $\beta(\mathbf{x}(t))$  and  $\mathbf{r}(t)$  and rearranging yields  $\|\dot{\mathbf{r}}(t)\| \cos \theta_r(t) < -\frac{\gamma}{D}$ . Thus, the component of  $\dot{\mathbf{r}}(t)$  in the direction of  $\mathbf{r}(t)$  has a negative sign and is upper bounded by a negative constant. This implies that  $\|\mathbf{r}(t)\|$  will decrease to a distance  $D$  in finite time.  $\square$

**Lemma 3.** If the agent enters the closed disk of diameter  $D$ , as described by Case II of Definition 1, then the agent will remain within the disk for all time.

*Proof.* Consider the case that  $\|\mathbf{r}(t)\| = D$  in (23). To exit the disk at a time  $t_1$ , the agent must satisfy  $\mathbf{r}(t_1) \cdot \dot{\mathbf{r}}(t_1) > 0$ . However, by continuity of  $\mathbf{x}(t)$ , there exists some  $\epsilon > 0$  such that  $\|\mathbf{r}(t_1 + \epsilon)\| > D$  and  $\mathbf{r}(t_1 + \epsilon) \cdot \dot{\mathbf{r}}(t_1 + \epsilon) > 0$ . This is infeasible by Definition 1, thus the agent will remain within the closed disk for all time.  $\square$

**Lemma 4.** If the agent is travelling along the constrained arc described by Definition 1, and transitions from Case I to Case II at a time  $t_1$  and distance  $\|\mathbf{r}(t_1)\| = D$ , then the control input is continuous at the transition.

*Proof.* When the agent transitions from Case I to Case II at  $t_1$ , we have  $\dot{\mathbf{r}}(t_1^-) \cdot \mathbf{r}(t_1^-) = -\beta(\mathbf{x}(t_1^-)) < 0$ . Continuity of  $\mathbf{x}(t)$  and  $\mathbf{x}_r(t)$  implies that  $\dot{\mathbf{r}}(t_1^+) \cdot \mathbf{r}(t_1^+) = -\beta(\mathbf{x}(t_1^+)) < 0$ . To stay on the constrained arc requires that  $\dot{\mathbf{r}}(t_1^+) \cdot \mathbf{r}(t_1^+) = 0$ . Thus, the agent must exit the constrained arc at  $t_1$ , and  $\mathbf{u}(t_1)$  is continuous by Corollary 1.  $\square$

By Lemmas 2–4 we have proven that our proposed barycentric motion constraint 1) drives the agent within a disk of diameter  $D$  in finite time, 2) traps the agent within the disk for all future time, and 3) does not introduce discontinuities into the agent’s trajectory. Next, we describe the constrained motion primitive for Case I of the barycentric motion constraint.

### B. Constrained Motion Primitive

To solve for the constrained motion of the agent when  $\|\mathbf{r}(t)\| > D$ , we use Hamiltonian analysis [14]. First, we construct the vector of tangency conditions,

$$\mathbf{N}(\mathbf{x}(t), t) = \left[ \beta(\mathbf{x}) + \mathbf{r}(t) \cdot \dot{\mathbf{r}}(t) \right], \quad (27)$$

and we append the derivative of (27) to the Hamiltonian,

$$H = \frac{1}{2} \|\mathbf{u}\|^2 + \boldsymbol{\lambda}^p \cdot \mathbf{v} + \boldsymbol{\lambda}^v \cdot \mathbf{u} + \mu \left( \dot{\beta} + \mathbf{r} \cdot \ddot{\mathbf{r}} + \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} \right). \quad (28)$$

The Euler-Lagrange equations are

$$\mathbf{u}(t) = -\boldsymbol{\lambda}^v(t) - \mu(t) \left( \dot{\beta}_u(\mathbf{x}(t)) + \mathbf{r}(t) \right), \quad (29)$$

$$-\dot{\boldsymbol{\lambda}}^v(t) = \boldsymbol{\lambda}^p(t) + \mu(t) \left( \dot{\beta}_v(\mathbf{x}(t)) + \dot{\mathbf{r}}(t) \right), \quad (30)$$

$$-\dot{\boldsymbol{\lambda}}^p(t) = \mu(t) \left( \dot{\beta}_p(\mathbf{x}(t)) + \ddot{\mathbf{r}}(t) \right). \quad (31)$$

To solve (29) - (31) we follow the method outlined in [12]. Since Problem 1 is a generalization of the problem reported in [12], we impose that  $\|\dot{\mathbf{r}}(t)\|$  is a constant. This is the *reigning optimal* solution [16] for this constrained motion primitive, which sacrifices solution optimality for an analytically tractable set of ordinary differential equations.

**Definition 2.** Consider the basis of  $\mathbb{R}^2$  defined by the vectors

$$\hat{p}(\mathbf{x}(t)) = \frac{\mathbf{r}(t)}{\|\mathbf{r}(t)\|} = \frac{\mathbf{r}(t)}{b(t)}, \quad (32)$$

$$\hat{q}(\mathbf{x}(t)) = \frac{\dot{\mathbf{r}}(t)}{\|\dot{\mathbf{r}}(t)\|} = \frac{\dot{\mathbf{r}}(t)}{a}, \quad (33)$$

where  $a = \|\dot{\mathbf{r}}(t)\|$  and  $b(t) = \|\mathbf{r}(t)\|$ . This is a well defined basis for  $\mathbb{R}^2$  as long as  $a \neq 0$ ,  $b(t) \neq 0$ , and  $\hat{p}(\mathbf{x}(t)) \cdot \hat{q}(\mathbf{x}(t)) \neq 1$ .

For simplicity we will omit the dependence of the unit vectors  $\hat{p}(\mathbf{x}(t))$  and  $\hat{q}(\mathbf{x}(t))$  on  $\mathbf{x}(t)$  when no ambiguity arises. To guarantee that the basis in Definition 2 is always well defined we select the following functional form for  $\beta$ ,

$$\beta(\mathbf{x}(t)) = a b(t) \kappa, \quad (34)$$

where  $\kappa \in (0, 1)$  is the cosine of the angle between  $\mathbf{r}$  and  $\dot{\mathbf{r}}$  by the definition of the dot product. The choice of (34) is a practical one, as it satisfies the premise of Lemma 2, it guarantees  $\hat{p}$  and  $\hat{q}$  are never colinear, and it is directly applicable when agents make bearing-only measurements [17]. As  $b(t) > D$ , we impose  $a > 0$  when traveling along the barycentric spiral to ensure  $\hat{p}$  and  $\hat{q}$  are well-defined.

Following the procedure of [12], we may project  $\ddot{\mathbf{r}}(t)$  onto the unit vectors  $\hat{p}$  and  $\hat{q}$ , which yields

$$\ddot{\mathbf{r}}(t) \odot \begin{bmatrix} \hat{p} \\ \hat{q} \end{bmatrix} = \begin{bmatrix} -\frac{a^2 + \dot{\beta}(\mathbf{x}(t))}{b(t)} \\ \dot{a} \end{bmatrix}, \quad (35)$$

where  $\odot$  is the component-wise inner product. Next we seek the time derivatives of (32) and (33). First,

$$\dot{\hat{p}} = \frac{\dot{\mathbf{r}}(t)}{b(t)} - \frac{\dot{b}(t)}{b(t)^2} \mathbf{r}(t) = \frac{a}{b(t)} \hat{q} - \frac{\dot{b}(t)}{b(t)} \hat{p}, \quad (36)$$

where

$$\begin{aligned} \dot{b}(t) &= \frac{d}{dt} \|\mathbf{p}(t) - \mathbf{p}_j(t)\| \\ &= \frac{\mathbf{p}(t) - \mathbf{p}_j(t)}{\|\mathbf{p}(t) - \mathbf{p}_j(t)\|} \cdot (\mathbf{v}(t) - \mathbf{v}_j(t)) \\ &= \frac{\mathbf{r}(t) \cdot \dot{\mathbf{r}}(t)}{b(t)} = \frac{-\beta(\mathbf{x}(t), t)}{b(t)} = -a\kappa, \end{aligned} \quad (37)$$

thus,

$$\dot{\hat{p}} = \frac{a}{b(t)} \hat{q} + \frac{a}{b(t)} \kappa \hat{p}. \quad (38)$$

It follows that

$$\dot{\hat{q}} = \frac{\ddot{\mathbf{r}}(t)}{a}, \quad (39)$$

and substituting (35) yields

$$\dot{\hat{q}} = -\frac{1}{ab(t)} \left( a^2 + \dot{\beta}(\mathbf{x}(t)) \right) \hat{p}, \quad (40)$$

where

$$\dot{\beta}(\mathbf{x}(t)) = a \dot{b}(t) \kappa = -(a\kappa)^2 = \dot{\beta}, \quad (41)$$

thus,

$$\dot{\hat{q}} = -\frac{a}{b(t)} (1 - \kappa^2) \hat{p}. \quad (42)$$

Substituting (34) into (35), and by the definition of  $\ddot{\mathbf{r}}$ , we have

$$(\mathbf{u}(t) - \mathbf{u}_r(t)) \odot \begin{bmatrix} \hat{p} \\ \hat{q} \end{bmatrix} = \begin{bmatrix} -\frac{a^2(1-\kappa^2)}{b(t)} \\ 0 \end{bmatrix}. \quad (43)$$

Next, we solve each row of (43) by substituting in the Euler-Lagrange equations and taking time derivatives until we have a system of ordinary differential equations that are only a function of  $a$ ,  $b(t)$ ,  $\mu(t)$ , and their derivatives. We start by decomposing (43) into a system of two equations,

$$(\mathbf{u}(t) - \mathbf{u}_r(t)) \cdot \hat{p} = -\frac{a^2}{b(t)}(1 - \kappa^2), \quad (44)$$

$$(\mathbf{u}(t) - \mathbf{u}_r(t)) \cdot \hat{q} = 0, \quad (45)$$

where substituting (29) and rearranging yields

$$(\boldsymbol{\lambda}^v(t) + \mathbf{u}_r(t) + \mu(t)\mathbf{r}(t)) \cdot \hat{p} = \frac{a^2}{b(t)}(1 - \kappa^2), \quad (46)$$

$$(\boldsymbol{\lambda}^v(t) + \mathbf{u}_r(t) + \mu(t)\mathbf{r}(t)) \cdot \hat{q} = 0, \quad (47)$$

which, by (27), simplifies to

$$(\boldsymbol{\lambda}^v(t) + \mathbf{u}_r(t)) \cdot \hat{p} = \frac{a^2}{b(t)}(1 - \kappa^2) - \mu(t)b(t), \quad (48)$$

$$(\boldsymbol{\lambda}^v(t) + \mathbf{u}_r(t)) \cdot \hat{q} = \mu(t)b(t)\kappa. \quad (49)$$

The next step is to take a time derivative of (48) and (49), then substitute (48) and (49) in for the  $\hat{p}$  and  $\hat{q}$  terms that appear. Then we substitute (30) into the resulting equations and simplify, which yields

$$(\dot{\mathbf{u}}_r(t) - \boldsymbol{\lambda}^p(t)) \cdot \hat{p} = 2a\mu(t)\kappa^2 - \dot{\mu}(t)b(t), \quad (50)$$

$$(\dot{\mathbf{u}}_r(t) - \boldsymbol{\lambda}^p(t)) \cdot \hat{q} = \frac{a^3}{b(t)^2}(1 - \kappa^2)^2 + \dot{\mu}(t)b(t)\kappa. \quad (51)$$

Finally, we take a time derivative of (50) and (51) and substitute (31). Applying Assumption 3 and simplifying yields

$$\frac{a^4}{b(t)^3}(1 - \kappa^2)^2 + \ddot{\mu}(t)b(t) = \frac{a^2}{b(t)}\mu(t)(1 - \kappa^2) + \mu(t)a\kappa, \quad (52)$$

$$a\dot{\mu}(t)(1 - \kappa^2) + \dot{\mu}(t)a\kappa = \ddot{\mu}(t)b(t)\kappa + 2\frac{a^4}{b(t)^3}(1 - \kappa^2)^2. \quad (53)$$

Equations (52) and (53) describe the evolution of  $\dot{\mu}(t)$  and  $b(t)$  for a given constant speed  $a$  and barycentric parameter  $\kappa$ .

To find the optimal control input to the agent we may integrate (37),

$$b(t) = b^0 - a\kappa(t - t^0), \quad (54)$$

where  $b(t_1) = b^0$ . Finally, substituting (41) and (54) into (35) yields

$$\ddot{\mathbf{r}}(t) = -\frac{a^2}{b_i^0 - a\kappa(t - t^0)}(1 - \kappa^2)\hat{p} + 0\hat{q}. \quad (55)$$

Let  $\theta_{pq}$  be the angle between  $\hat{p}$  and  $\hat{q}$ , then, by definition of the dot product,

$$\begin{aligned} \ddot{\mathbf{r}}(t) \cdot \hat{p} &= \|\ddot{\mathbf{r}}(t)\| \|\hat{p}\| \cos\left(\frac{\pi}{2} - \theta_{pq}\right) = \|\ddot{\mathbf{r}}(t)\| \sin(\theta_{pq}) \\ &= \|\ddot{\mathbf{r}}(t)\| \sin(\arccos(\kappa)) = \|\ddot{\mathbf{r}}(t)\| \sqrt{1 - \kappa^2}. \end{aligned} \quad (56)$$

Thus,

$$\|\ddot{\mathbf{r}}(t)\| = \frac{a^2}{b_i^0 - a\kappa(t - t^0)} \sqrt{1 - \kappa^2}, \quad (57)$$

and the orientation of  $\ddot{\mathbf{r}}(t)$  is perpendicular to  $\hat{q}$ . In the next subsection we use (52) and (53) to determine how the agent will optimally transition from the unconstrained to the constrained arc.

### C. Barycentric Jump Conditions

Let the agent transition from an unconstrained to barycentric-constrained arc at some time  $t$ . The jump conditions for the influence functions are [14],

$$\boldsymbol{\lambda}_p(t^-) = \boldsymbol{\lambda}_p(t^+) + a\pi[\kappa\hat{p} + \hat{q}], \quad (58)$$

$$\boldsymbol{\lambda}_v(t^-) = \boldsymbol{\lambda}_v(t^+) + b(t)\pi[\hat{p} + \kappa\hat{q}]. \quad (59)$$

Substituting (29) into (59) and applying continuity of  $\mathbf{u}(t)$  implies

$$\mu(t^+)\hat{p} = \pi[\hat{p} + \kappa\hat{q}]. \quad (60)$$

Next, we project (60) onto the unit vectors  $\hat{p}$  and  $\hat{q}$  which yields two scalar equations,

$$\mu(t^+) = \pi[1 - \kappa^2], \quad (61)$$

$$-\kappa\mu(t^+) = \pi[-\kappa + \kappa]. \quad (62)$$

By definition  $\kappa \in (0, 1)$ , which implies that  $\mu(t^+) = 0$  and  $\pi = 0$ . Thus we may simplify (58) to

$$\boldsymbol{\lambda}_p(t^-) = \boldsymbol{\lambda}_p(t^+). \quad (63)$$

Finally, the time derivatives of  $\mathbf{u}(t)$  are

$$\dot{\mathbf{u}}(t^-) = \boldsymbol{\lambda}_p(t^-), \quad (64)$$

$$\dot{\mathbf{u}}(t^+) = \boldsymbol{\lambda}_p(t^+) + \dot{\mu}(t^+)\mathbf{r}(t), \quad (65)$$

thus

$$\dot{\mathbf{u}}(t^+) - \dot{\mathbf{u}}(t^-) = \dot{\mu}(t^+)\mathbf{r}(t). \quad (66)$$

We may substitute  $\mu(t^+) = 0$  into (52) and (53), which yields

$$\dot{\mu}(t^+) = \frac{a^3}{b(t)^3} \kappa \frac{(1 - \kappa^2)^2}{1 - \kappa^2 + \kappa}. \quad (67)$$

Thus,

$$\dot{\mathbf{u}}(t^+) - \dot{\mathbf{u}}(t^-) = \mathbf{r}(t) \left( \frac{a^3}{b(t)^3} \kappa \frac{(1 - \kappa^2)^2}{1 - \kappa^2 + \kappa} \right). \quad (68)$$

At the junction between the unconstrained and constrained cases, we have four unknowns,  $\mathbf{p}(t)$ ,  $\|\mathbf{v}(t)\|$ , and the optimal transition time  $t$ . The corresponding four equations are continuity in  $\mathbf{u}(t)$ , by Theorem 1, and (68). The velocity  $\mathbf{v}(t)$  is determined by solving  $\mathbf{N}(\mathbf{x}(t), t) = \mathbf{0}$  via (27). In the next section we provide the optimality conditions when activating the disk constraint given in Case II of Definition 1, which also applies to collision avoidance.

#### D. Fixed Distance Constraint

When  $\|\mathbf{r}(t)\| \leq D$  the tangency equations are

$$\mathbf{N}(\mathbf{x}(t), t) = \begin{bmatrix} \mathbf{r}(t) \cdot \mathbf{r}(t) - D^2 \\ 2\mathbf{r}(t) \cdot \dot{\mathbf{r}}(t) \end{bmatrix}, \quad (69)$$

$$g^{(2)}(\mathbf{x}(t), t) = 2\mathbf{r}(t) \cdot \ddot{\mathbf{r}}(t) + 2\dot{\mathbf{r}}(t) \cdot \dot{\mathbf{r}}(t), \quad (70)$$

which leads to the same analysis as Section III when  $\mathbf{N}(\mathbf{x}(t), t) = \mathbf{0}$  and  $g^{(2)}(\mathbf{x}(t), t) = 0$  if we impose  $b(t) = D$  and  $\kappa = 0$ . In this case (53) implies that  $\mu(t)$  is a constant, and (52) implies that

$$\mu = \left(\frac{a}{D}\right)^2. \quad (71)$$

The derivative of  $\mathbf{N}$  with respect to the state is

$$\frac{\partial \mathbf{N}(\mathbf{x}(t), t)}{\partial \mathbf{x}(t)} = \begin{bmatrix} 2\mathbf{r}(t)^T & 0 \\ 2\dot{\mathbf{r}}(t)^T & 2\mathbf{r}(t)^T \end{bmatrix}, \quad (72)$$

and the jump conditions at time  $t$  are [14],

$$\boldsymbol{\lambda}_p(t^-) = \boldsymbol{\lambda}_p(t^+) + 2[\pi_1 \mathbf{r}(t) + \pi_2 \dot{\mathbf{r}}(t)], \quad (73)$$

$$\boldsymbol{\lambda}_v(t^-) = \boldsymbol{\lambda}_v(t^+) + 2[\pi_2 \mathbf{r}(t)], \quad (74)$$

The condition  $\frac{\partial H}{\partial \mathbf{u}}|_{t^-} = \frac{\partial H}{\partial \mathbf{u}}|_{t^+}$  implies

$$\mathbf{u}(t^-) + \boldsymbol{\lambda}_v(t^-) = \mathbf{u}(t^+) + \boldsymbol{\lambda}_v(t^+) + \mu(t^+) \mathbf{r}(t), \quad (75)$$

which, by Theorem 1, implies

$$\boldsymbol{\lambda}_v(t^-) = \boldsymbol{\lambda}_v(t^+) + \mu(t^+) \mathbf{r}(t). \quad (76)$$

Thus, by (74),

$$\pi_2 = \frac{\mu(t^+)}{2} = \frac{a^2}{2D^2}. \quad (77)$$

Combining the time derivative of (29) with (30) and substituting it along with (77) into (73) implies  $\dot{\mathbf{u}}(t^-) = \dot{\mathbf{u}}(t^+) + 2D\pi_1 \mathbf{r}(t)$ , which, projected onto  $\hat{q}$  yields

$$\dot{\mathbf{u}}(t^-) \cdot \hat{q} = \dot{\mathbf{u}}(t^+) \cdot \hat{q}. \quad (78)$$

We have three unknowns: the angle of  $\mathbf{r}(t)$ , the relative speed  $a = \|\dot{\mathbf{r}}(t)\|$ , and the optimal transition time  $t$ . The three corresponding equations are continuity in  $\mathbf{u}(t)$  by Theorem 1 and the continuity of  $\dot{\mathbf{u}}(t) \cdot \hat{q}$  by (78). This is sufficient to determine the agent's state using  $\mathbf{N}(\mathbf{x}(t), t) = \mathbf{0}$  via (69).

Finally, we note that in a swarm system, for two agents  $i, j \in \mathbb{N}$ , we may write a safe distance constraint for agent  $i$  relative to agent  $j$ ,

$$g_{ij}(\mathbf{x}(t), t) = (2R)^2 - \|\mathbf{p}(t) - \mathbf{p}_j(t)\|^2 \leq 0, \quad (79)$$

for some agent radius  $R \in \mathbb{R}_{>0}$ . This results in tangency conditions identical to (69) when the constraint is active, i.e.,  $\mathbf{N}(\mathbf{x}(t), t) = \mathbf{0}$ . Thus, the preceding analysis holds for the transition to the collision avoidance constraint if  $j$ 's trajectory satisfies Assumption 3.

#### V. CONCLUSIONS AND FUTURE WORK

In this paper, we presented a set of optimality conditions for a class of energy-minimizing systems when transitioning between constrained and unconstrained trajectories. These conditions only require information about the agent's state and control dynamics, and we presented a case study for them using an original barycentric motion constraint. We extended this analysis to include collision avoidance.

Ongoing research addresses the potential of deriving an efficient shooting method to numerically solve the proposed jump conditions in real-time. Future work should consider extending Theorem 1 for the cases where (1) only the right partial derivatives of  $\mathbf{N}(\mathbf{x}(t), t)$  exist, and (2) a constraint becomes active only at a single instant in time. Finding optimality conditions for a sequence of constraint activations is another interesting research direction.

#### REFERENCES

- [1] H. Oh, A. R. Shirazi, C. Sun, and Y. Jin, "Bio-inspired self-organising multi-robot pattern formation: A review," *Robotics and Autonomous Systems*, vol. 91, pp. 83–100, 2017.
- [2] B. Chalaki, L. E. Beaver, and A. A. Malikopoulos, "Experimental validation of a real-time optimal controller for coordination of cavs in a multi-lane roundabout," in *31st IEEE Intelligent Vehicles Symposium (IV)*, 2020, pp. 504–509.
- [3] Q. Lindsey, D. Mellinger, and V. Kumar, "Construction with quadrotor teams," *Autonomous Robots*, 2012.
- [4] J. Cortes, "Global formation-shape stabilization of relative sensing networks," in *Proceedings of the American Control Conference*, 2009.
- [5] D. Pickem, P. Glotfelter, L. Wang, M. Mote, A. Ames, E. Feron, and M. Egerstedt, "The Robotarium: A remotely accessible swarm robotics research testbed," in *IEEE International Conference on Robotics and Automation*, 7 2017, pp. 1699–1706.
- [6] M. Rubenstein, C. Ahler, and R. Nagpal, "Kilobot: A low cost scalable robot system for collective behaviors," in *Proceedings of the 2012 IEEE International Conference on Robotics and Automation*, 2012.
- [7] G. Vásárhelyi, C. Virágh, G. Somorjai, T. Nepusz, A. E. Eiben, and T. Vicsek, "Optimized flocking of autonomous drones in confined environments," *Science Robotics*, vol. 3, no. 20, 2018.
- [8] A. A. Malikopoulos and L. Zhao, "A closed-form analytical solution for optimal coordination of connected and automated vehicles," in *2019 American Control Conference (ACC)*. IEEE, 2019, pp. 3599–3604.
- [9] A. A. Malikopoulos, L. E. Beaver, and I. V. Chremos, "Optimal time trajectory and coordination for connected and automated vehicles," *Automatica*, vol. 125, p. 109469, 2021.
- [10] L. E. Beaver and A. A. Malikopoulos, "Beyond Reynolds: A Constraint-Driven Approach to Cluster Flocking," in *IEEE 59th Conference on Decision and Control*, 2020, pp. 208–213.
- [11] A. I. Mahbub, A. A. Malikopoulos, and L. Zhao, "Decentralized optimal coordination of connected and automated vehicles for multiple traffic scenarios," *Automatica*, vol. 117, p. 108958, 2020.
- [12] L. E. Beaver, C. Kroninger, and A. A. Malikopoulos, "An Optimal Control Approach to Flocking," in *2020 American Control Conference*, 2020, pp. 683–688.
- [13] K. Fathian, S. Safaoui, T. H. Summers, and N. R. Gans, "Robust 3D Distributed Formation Control with Collision Avoidance and Application to Multirotor Aerial Vehicles," in *2019 International Conference on Robotics and Automation*, 2019, pp. 9209–9215.
- [14] A. E. J. Bryson and Y.-C. Ho, *Applied Optimal Control: Optimization, Estimation, and Control*. John Wiley and Sons, 1975.
- [15] G. Notomista and M. Egerstedt, "Constraint-Driven Coordinated Control of Multi-Robot Systems," in *Proceedings of the 2019 American Control Conference*, 2019.
- [16] I. M. Ross, *A Primer on Pontryagin's Principle in Optimal Control*, 2nd ed., E. Solon, Ed. San Francisco: Collegiate Publishers, 2015.
- [17] A. Boccia, A. Adalo, D. V. Dimarogonas, M. di Bernardo, and K. H. Johansson, "Tracking a mobile target by multi-robot circumnavigation using bearing measurements," in *IEEE 56th Annual Conference on Decision and Control*. IEEE, 2017, pp. 1076–1081.